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TWO-WAY ANALYSIS OF VARIANCE FOR WEIBULL POPULATIONS.(U)
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Techniques and software are developed for the analysis of experiments in which the response variable follows the two-parameter and three-parameter Weibull distribution.

For the two-parameter case the analogue of the analysis of variance is considered in which the hypothesis that the scale parameter is affected by the levels of an external factor is tested by a shape parameter ratio test. (Continued on page two) next page

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20. Abstract continued.

The analogue of Weibull regression is also treated for the case where the scale parameter varies as a power function of a quantitative external factor.

A new method is found for drawing inferences on the location parameter of the three-parameter Weibull model, based on maximum likelihood shape parameter estimates. The sample sizes and censoring numbers for which empirical distribution have been determined are summarized.

A summary is given of the software developed. Several applications of the techniques are also summarized.

FINAL SCIENTIFIC REPORT

ON

TWO-WAY ANALYSIS OF VARIANCE FOR WEIBULL POPULATIONS

DECEMBER 1978

Prepared: John I. McCool

Approved:

SKF REPORT NO. AL78P046

Submitted to:

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TWO-WAY ANALYSIS OF VARIANCE FOR WEIBULL POPULATIONS

1.0 SINGLE CLASSIFICATION EXPERIMENTS

During Fiscal Year 1978 the work begun in 1977 on the analysis of single classification experiments was brought to full fruition with the completion of a comprehensive (50 pages) report containing theory, tables and worked examples of both the analysis and design of single classification experiments with two-parameter Weibull variates. The report has now been accepted for publication by the Journal of Statistical Planning and Inference [1].

Extensive additional tables of the numerical values needed for analyzing single classification Weibull experiments were also computed. Approximately 35 hours of CPU time on SKF's IBM 370/158 computer were used in generating these tables. A small number of sets have been reproduced together with the report discussed above as an SKF publication and will be available to qualified requestors. Copies have been sent to the Rome Air Development Center for the use of the reliability analysts headquartered at that site. Copies have also been distributed throughout the SKF world-wide organization with a cover letter citing the potential for savings in bearing testing that follow from adoption of these methods.

The methods for analyzing single and crossed classification experiments developed under this contract have been successfully applied in other DOD sponsored work performed at SKF. For the Navy, under contract N00019-76-C-0168, we have examined alternative plans for endurance testing three varieties of silicon nitride material in rolling contact.

For the Army, under Contract DAAK70-77-C-0034 we have compared the effect of six types of grease on the endurance performance of tapered roller bearings.

In a further effort to disseminate our results, a presentation was made to six staff members of the Reliability Branch of Rome Air Development Center at Rome, N.Y. on March 29, 1978. As a consequence of that meeting life test data on integrated circuits were communicated to us by RADC/RBR for analysis. The data were found to have been collected in grouped form rather than as individual failure times and were not therefore amenable to the analysis developed under this contract. A more limited nonparameter analysis was performed and communicated to RADC in the form of a short report [2].

We have presented a talk based on the single classification analysis entitled "The Comparative Power of the Likelihood Ratio and a Shape Parameter Ratio Test for the Equality of Weibull Scale Parameters" at the Joint Statistical Meetings in San Diego, Calif., on August 17, 1978. A copy of the documentation distributed at the meeting is given in Appendix I.

Two important new areas of investigation have been opened up during this contract year, namely, 1) Weibull regression and 2) location parameter estimation and inference.

2.0 WEIBULL REGRESSION

The regression problem is applicable to situations where life tests are conducted at each of several levels of a variable arbitrarily termed a "stress". In applications the "stress" could be a voltage, temperature, load, etc. The Weibull scale parameter or "characteristic" life is assumed to vary inversely with a power of the stress.

A numerical scheme has been developed for the joint estimation by the method of maximum likelihood of the Weibull shape parameter and the stress-life exponent.

The method has been implemented in a computer program "REGSIM" to produce, using Monte Carlo methods, the distribution of the statistical quantities needed for setting confidence limits on the Weibull shape parameter, the load-life exponent and a fixed percentage point of the life distribution at any desired stress level. The stress levels at which inferences are to be made may be outside the range of stresses at which life tests were conducted. The results are thus applicable to the analysis of "accelerated" tests, a common practice in the engineering and physical sciences.

In addition to REGSIM a program called "REGEST" has been developed to perform estimation on sets of data taken at specified stress levels. A paper has been prepared entitled "Inference in Weibull Regression". It contains theoretical details and a full numerical illustration of the application of the techniques. A copy is included as Appendix II.

3.0 LOCATION PARAMETER ESTIMATION AND INFERENCE

The Weibull location or threshold parameter is, in the life testing context, the value prior to which failure cannot take place. In ordnance applications the threshold parameter might represent the time (after arming) before which a bomb cannot explode.

It is important to be able to estimate and set a lower bound for the location parameter based on a set of observations. We have during the current contract year explored and found feasible, a technique for estimating and setting lower bounds on the Weibull scale parameter. A program called LOCEST was developed to perform this analysis on a set of data. The program LOCSIM develops, by Monte Carlo sampling, the tabular data needed for implementing the estimation procedure and optimally selecting the early order statistic used in the calculation.

Several production runs have been made using LOCSIM to try to determine a rule for choosing the early order statistic at which to make an estimate of the Weibull slope for best (most powerful) detection of a non zero location parameter. The fifth order statistic works best in all cases examined. Below $r_1=5$ overflow problems were encountered in the iterative estimation routines caused by occasional large values of the estimated shape parameter. We have incorporated the closed form expression for the ML shape parameter estimate applicable when censoring at the second failure. This permits trouble free solution at $r_1=2$ and confirms that there is an optimum value of $r_1 > 2$. Thus the optimum r_1 has been demonstrated to satisfy $2 < r_1 \leq 5$ in all cases examined.

Null and non-null runs have now been made in which the distribution of the test statistic for evaluating the location parameter was determined with $r_1=5$ for various sample sizes (n) and censoring amounts (r), specifically: n=10, r=6,7,8,9,10, n=20, r=5,10,12,15,18,20, n=25, r=10,14,18,20,25, n=30, r=10,15, 20,25,30.

An additional set of runs was made with $r_1=2$ and n=10, r=5,3,6,7,8,10, n=15, r=5,10,11,12,15.

This is sufficient information to warrant the preparation of of a paper. This will be undertaken early in the next contract year.

A summary of the software developed to date under this contract follows.

SOFTWARE SUMMARY

PROGRAM NAME	FUNCTION
WEIBEST	Analyzes k sets of Weibull data. Computes individual shape parameters, 3 percentiles and printer plot. Calculates shape parameter ratio test statistic, likelihood ratio test statistic and ratio of largest to smallest individual group shape parameter estimates.
WEIBSIM	Calculates critical values of SPR and LR test statistics and distributions for interval estimation of percentiles and shape parameter.
REGEST	Calculates exponent of power function relation between scale parameter and stress, 3 percentage points at each stress level and the shape parameter.
REGSIM	Simulates the distribution of 5 pivotal functions needed for setting single and joint confidence intervals in Weibull regression.
FACSIM	Calculates the distribution of functions required for testing significance of row and column effects in factorial experiment with no interaction.
LOCEST	Calculates the test statistic for location parameter and the median unbiased and lower 95% limit for location parameter.
LOCSIM	Computes null distribution of shape parameter estimates based on first r_1 and r order statistics in samples of size n.

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1. McCool, J. I., "Analysis of Single Classification Experiments Based on Censored Samples from the Two-Parameter Weibull Distribution," accepted for publication in the Journal of Statistical Planning and Inference.

2. McCool, J. I., "Analysis of RADC Life Test Data," SKF Report No. AL78P003L, July 1978.

APPENDIX I

THE COMPARATIVE POWER OF THE LIKELIHOOD RATIO AND A SHAPE PARAMETER RATIO TEST FOR THE EQUALITY OF WEIBULL SCALE PARAMETERS

John I. McCool

August, 1978

SKF REPORT NO. AL78P025

This material was prepared for distribution at the 1978 Joint Statistical Meetings, San Diego, California, August 14-17, 1978.

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THE COMPARATIVE POWER OF THE LIKELIHOOD RATIO AND A SHAPE PARAMETER RATIO TEST FOR THE EQUALITY OF WEIBULL SCALE PARAMETERS

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1. INTRODUCTION

We consider k two-parameter Weibull populations having a common shape parameter β and possibly different scale parameters $\eta_i \ldots \eta_k$. The CDF of the i-th population is written:

$$F(x) = 1 - \exp\{-[x/\eta_i]^{\beta}\} \tag{1}$$

Given samples of size n from each population, each type II censored at the r-th failure, the maximum likelihood estimate of β when the η_i are presumed to differ, is the solution $\hat{\beta}_1$ of:

$$1/\hat{\beta}_1 + \frac{1}{rk}\sum_{i=1}^{k}\sum_{j=1}^{r}\log x_{i(j)} - \frac{1}{k}\sum_{i=1}^{k}\left[\sum_{j=1}^{n} x_{i(j)}^{\hat{\beta}_1}\log x_{i(j)} \Big/ \sum_{j=1}^{n} x_{i(j)}^{\hat{\beta}_1}\right] = 0 \tag{2}$$

where $x_{i(j)}$ is the j-th order statistic in the sample from the i-th population.

The ML estimate of η_i is

$$\hat{\eta}_i = \left\{\frac{1}{r}\sum_{j=1}^{n} x_{i(j)}^{\hat{\beta}_1}\right\}^{1/\hat{\beta}_1} \tag{3}$$

When the populations have a common scale parameter $\eta_i = \eta_o$, the ML shape parameter estimate is the solution $\hat{\beta}_o$ of:

$$1/\hat{\beta}_o + \frac{1}{rk}\sum_{i=1}^{k}\sum_{j=1}^{r}\log x_{i(j)} - \frac{\sum_{i=1}^{k}\sum_{j=1}^{n} x_{i(j)}^{\hat{\beta}_o}\log x_{i(j)}}{\sum^{k}\sum^{n} x_{i(j)}^{\hat{\beta}_o}} = 0 \tag{4}$$

The ML estimate of η_o is

$$\hat{\eta}_o = \left\{\sum_{i=1}^{k}\sum_{j=1}^{n} x_{i(j)}^{\hat{\beta}_o}/rk\right\}^{1/\hat{\beta}_o} \tag{5}$$

2. TESTING EQUALITY OF SCALE PARAMETERS

We consider two tests of the hypothesis $H_o: \eta_i = \eta_o$ a shape parameter ratio (SPR) test based on the ratio $\hat{\beta}_1/\hat{\beta}_0$ of the two shape parameter estimates of Eqs. (2) and (4) and a test based on the likelihood ratio λ using as test statistic the monotonically transformed value $-2\log\lambda$ expressible as,

$$-2\log\lambda = 2rk\log(\hat{\beta}_1/\hat{\beta}_0) + 2\sum_{i=1}^{k}\sum_{j=1}^{r}\left[\log\left(\frac{x_{i(j)}}{\hat{\eta}_i}\right)^{\hat{\beta}_1} - \log\left(\frac{x_{i(j)}}{\hat{\eta}_o}\right)^{\hat{\beta}_0}\right] \tag{6}$$

The distributions of both test statistics depend only upon n, r and k under the null hypothesis.

Table 1 gives 90 and 95% points of $\hat{\beta}_1/\hat{\beta}_0$ and $-2\log\lambda$ determined by Monte Carlo sampling for various n, r and k with n ranging from 5 to 30 and k, from 2 to 10.

The degree to which $-2\log\lambda$ approaches the asymptotic χ^2 distribution based on k-1 degrees of freedom may be judged by their respective 95-th percentiles, listed in columns 7 and 8 of Table 1.

3. POWER OF HYPOTHESIS TESTS

The power of the SPR test against various alternatives has been found to depend only upon the value of the symmetric function of the η_i:

$$\phi_1 = \sum_{i=1}^{k}\log\eta_i^{\beta}\,[\eta_i^{\beta}/\sum_{i=1}^{k}\eta_i^{\beta} - {}^{1}/k] \tag{7}$$

ϕ_1 is scale invariant. When the η_i's are equal, $\phi_1 = 0$, and when the η_i are not all equal, $\phi_1 > 0$.

The power of the LR test depends on ϕ_1 and another symmetric, scale invariant, postive function of the η_i's given by

$$\phi_2 = \sum_{i=1}^{k}\log\,[\sum_{i=1}^{k}\eta_i^{\beta}/k\eta_i^{\beta}] \tag{8}$$

For equal ϕ_1 the power of the LR test may be slightly inferior or slightly superior to the SPR test, depending upon the value of ϕ_2. In general, SPR is best against a single aberrant i.e., when all η_i's are equal but one, and k>2; the LR test is superior against more "diffuse" alternatives in which several η_i's differ.

For fixed r the power of both tests, against the single aberrant alternative, decreases with n and increases with k.

Figure 1 shows, for k=2 and n=30 the value of the acceptance probability plotted for various r against η_1^β (η_2^β=1.0). The SPR and LR tests are indistinguishable to graphical accuracy. These values may be used, for each r as a conservative approximation for the acceptance probabilities against the single aberrant alternative, when k>2 and n<30.

4. INTERVAL ESTIMATION OF THE SHAPE PARAMETER

The simulations that were used to find the critical values of the SPR and LR tests also yielded the distributions of the function

$$v \equiv \hat{\beta}_1/\hat{\beta} \tag{9}$$

This distribution depends only on n, r and k. A $100(1-\alpha)\%$ interval estimate of β is obtained by inverting the inequalities in the probability statement

$$\Pr\left[v_{\alpha/2} < \hat{\beta}_1/\beta < v_{1-\alpha/2}\right] = 1 - \alpha \tag{10}$$

For 90% confidence the interval is

$$\hat{\beta}_1/v_{0.95} < \beta < \hat{\beta}_1/v_{0.05} \tag{11}$$

The ratio R of the upper to lower end of this interval is given by

$$R = v_{0.95}/v_{0.05} \tag{12}$$

The quantity R may be used to characterize the precision within which β has been determined by the k censored samples of size n. Figure 1 shows some plots of R against censoring number for n=5, 15 and 30 and with k=1 and k=5. For a given value of r the precision worsens with n. For fixed n precision improves with r. Considerable precision is gained, particularly at low censoring numbers, if samples can be pooled for estimating β, i.e., if k>1.

5. INTERVAL ESTIMATION OF A PERCENTILE OF THE i-th POPULATION

The 100 p-th percentile of the i-th population is expressible in terms of η_i and β as

$$x_{p_i} = [-\log(1-p)]^{1/\beta} \cdot \eta_i \tag{13}$$

Its ML estimate is correspondingly

$$\hat{x}_{p_i} = [-\log(1-p)]^{1/\hat{\beta}_1} \cdot \hat{\eta}_i \tag{14}$$

The function

$$u \equiv \hat{\beta}_1 \log(\hat{x}_{p_i}/x_{p_i}) \tag{15}$$

follows a distribution that depends on n, r and k only. In terms of the 5-th and 95-th percentiles of u, a 90% confidence interval for x_{p_i} is calculable as

$$\hat{x}_{p_i} \cdot \exp[-u_{0.95}/\hat{\beta}] < x_{p_i} < \hat{x}_{p_i} \cdot \exp[-u_{0.05}/\hat{\beta}] \tag{16}$$

The ratio of the upper to lower ends of this confidence interval is a random variable through its dependence on $\hat{\beta}$. The β-th power of the median value of this random variable denoted may be expressed as

$$R^{\beta}_{0.50} = \exp[(-u_{0.05} + u_{0.95})/v_{0.50}] \tag{17}$$

and may be used as a measure of the precision with which x_{p_i} has been determined.

Figure 2 shows $R^{\beta}_{0.50}$ for estimating the tenth percentile $x_{0.10}$ of any population, as a function of censoring number for some n and k values. Again, pooling of samples believed to have a common shape parameter greatly increases the precision in the determination of $x_{0.10}$.

			$\hat{\beta}_1/\hat{\beta}_0$		$-2\log\lambda$		$\chi^2(k-1)$
n	r	k	0.90	0.95	0.90	0.95	0.95
5	3	2	1.799	2.144	5.357	7.235	3.84
5	3	3	1.788	2.078	7.913	10.24	5.99
5	3	4	1.769	2.009	10.29	12.80	7.81
5	3	5	1.744	1.925	12.34	15.07	9.49
5	3	10	1.644	1.747	21.78	24.88	16.9
5	5	2	1.259	1.366	3.794	5.290	3.84
5	5	3	1.273	1.361	5.984	7.804	5.99
5	5	4	1.275	1.353	8.076	10.08	7.81
5	5	5	1.267	1.339	9.843	12.10	9.49
5	5	10	1.246	1.284	17.94	20.72	16.9
10	5	2	1.363	1.508	4.142	5.782	3.84
10	5	3	1.372	1.491	6.443	8.372	5.99
10	5	4	1.367	1.463	8.467	10.51	7.81
10	5	5	1.365	1.438	10.39	12.45	9.49
10	5	10	1.328	1.378	18.84	21.58	16.9
10	10	2	1.105	1.150	3.165	4.472	3.84
10	10	3	1.116	1.149	5.317	6.834	5.99
10	10	4	1.115	1.142	6.981	8.706	7.81
10	10	5	1.114	1.139	8.755	10.55	9.49
10	10	10	1.105	1.121	16.16	18.61	16.9
15	5	2	1.374	1.535	4.090	5.770	3.84
15	5	3	1.392	1.522	6.393	8.428	5.99
15	5	4	1.392	1.491	8.579	10.63	7.81
15	5	5	1.377	1.462	10.23	12.44	9.49
15	10	2	1.131	1.190	3.260	4.653	3.84
15	10	3	1.144	1.185	5.370	6.846	5.99
15	10	4	1.145	1.181	7.224	8.962	7.81
15	10	5	1.143	1.176	8.881	10.87	9.49
15	15	2	1.066	1.096	3.101	4.436	3.84
15	15	3	1.072	1.095	5.071	6.562	5.99
15	15	4	1.074	1.091	6.843	8.497	7.81
15	15	5	1.074	1.090	8.477	10.38	9.49
20	5	2	1.382	1.546	4.110	5.749	3.84
20	5	3	1.408	1.542	6.455	8.469	5.99
20	5	4	1.398	1.495	8.487	10.51	7.81
20	5	5	1.389	1.478	10.34	12.41	9.49
20	10	2	1.139	1.197	3.220	4.587	3.84
20	10	3	1.155	1.202	5.453	7.004	5.99
20	10	4	1.155	1.195	7.233	9.120	7.81
20	10	5	1.155	1.190	9.068	11.01	9.49
20	15	2	1.079	1.112	3.088	4.297	3.84
20	15	3	1.087	1.115	5.102	6.583	5.99
20	15	4	1.088	1.111	6.862	8.591	7.81
20	15	5	1.089	1.108	8.603	10.56	9.49

Table 1. 90-th & 95-th Percentiles of SPR and LR Test Statistics

			$\hat{\beta}_1/\hat{\beta}_0$		$-2\log\lambda$		$\chi^2(k-1)$
n	r	k	0.90	0.95	0.90	0.95	0.95
20	20	2	1.048	1.068	3.012	4.194	3.84
20	20	3	1.054	1.069	4.986	6.456	5.99
20	20	4	1.054	1.068	6.740	8.460	7.81
20	20	5	1.053	1.065	8.428	10.23	9.49
25	5	2	1.383	1.545	4.003	5.634	3.84
25	5	3	1.400	1.522	6.365	8.142	5.99
25	5	4	1.402	1.498	8.518	10.40	7.81
25	5	5	1.390	1.481	10.29	12.42	9.49
25	10	2	1.143	1.199	3.211	4.485	3.84
25	10	3	1.160	1.208	5.403	7.038	5.99
25	10	4	1.161	1.202	7.274	9.074	7.81
25	10	5	1.158	1.191	8.872	10.83	9.49
25	15	2	1.084	1.122	3.079	4.465	3.84
25	15	3	1.094	1.123	5.116	6.633	5.99
25	15	4	1.096	1.123	6.992	8.885	7.81
25	15	5	1.095	1.117	8.544	10.52	9.49
25	20	2	1.056	1.078	2.994	4.250	3.84
25	20	3	1.063	1.081	4.988	6.453	5.99
25	20	4	1.063	1.080	6.783	8.448	7.81
25	20	5	1.062	1.077	8.212	10.15	9.49
25	25	2	1.037	1.054	2.945	4.210	3.84
25	25	3	1.042	1.054	4.894	6.450	5.99
25	25	4	1.042	1.053	6.616	8.330	7.81
25	25	5	1.041	1.051	8.122	9.919	9.49
30	5	2	1.394	1.571	4.111	5.852	3.84
30	5	3	1.414	1.544	6.489	8.396	5.99
30	5	4	1.411	1.529	8.601	10.83	7.81
30	5	5	1.399	1.492	10.37	12.67	9.49
30	10	2	1.148	1.212	3.259	4.704	3.84
30	10	3	1.165	1.215	5.452	7.200	5.99
30	10	4	1.167	1.206	7.402	9.171	7.81
30	10	5	1.165	1.199	9.071	11.03	9.49
30	15	2	1.087	1.124	3.033	4.317	3.84
30	15	3	1.096	1.125	5.046	6.549	5.99
30	15	4	1.099	1.125	6.967	8.663	7.81
30	15	5	1.098	1.121	8.588	10.51	9.49
30	20	2	1.059	1.083	2.921	4.167	3.84
30	20	3	1.065	1.084	4.844	6.256	5.99
30	20	4	1.067	1.084	6.687	8.405	7.81
30	20	5	1.067	1.082	8.361	10.17	9.49
30	25	2	1.044	1.061	2.956	4.116	3.84
30	25	3	1.047	1.061	4.753	6.168	5.99
30	25	4	1.048	1.060	6.562	8.082	7.81
30	25	5	1.048	1.058	8.128	9.886	9.49

Table 1 (Continued)

n	r	k	$\hat{\beta}_1/\hat{\beta}_0$ 0.90	$\hat{\beta}_1/\hat{\beta}_0$ 0.95	$-2\log\lambda$ 0.90	$-2\log\lambda$ 0.95	$\chi^2(k-1)$ 0.95
30	30	2	1.031	1.044	2.890	4.025	3.84
30	30	3	1.035	1.044	4.765	6.192	5.99
30	30	4	1.035	1.044	6.511	8.123	7.81
30	30	5	1.035	1.042	8.121	9.938	9.49

Table 1 (Continued)

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1. McCool, J. I., "Analysis of Variance for Weibull Populations", in the Theory and Applications of Reliability, edited by C. Tsokos and I. Shimi, Academic Press, (1977).

2. McCool, J. I., "Analysis of Single Classification Experiments Based on Censored Samples from the Two-Parameter Weibull Distribution", submitted for publication.

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APPENDIX II

INFERENCE IN WEIBULL REGRESSION

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SKF REPORT NO. AL78P045

NOVEMBER 1978

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INFERENCE IN WEIBULL REGRESSION

ABSTRACT

We consider an experimental situation in which a response variable follows a two-parameter Weibull distribution having a scale parameter that varies inversely with a power of a deterministic, externally controlled, variable generically termed a "stress". The shape parameter is presumed to be invariant with stress. Equations are formed, from the results of type II censored life tests conducted at each of several stresses, whose solution yield the maximum likelihood estimates of the common shape parameter, the stress-life exponent, and a general percentile of the life distribution applicable at an arbitrary stress level. A numerical scheme for solving the equations is given.

Pivotal functions of the ML estimates are found whereby interval and median unbiased point estimates may be calculated once the distribution of the pivotal functions is found by Monte Carlo sampling. A numerical example of the calculation of point and interval estimates is given. The precision with which the shape parameter is estimated by testing at several stresses is comparable to the precision applicable to a single test of the same total sample size. The precision in estimating percentiles is a maximum near the middle of the stress range at which testing was performed and is, at that point, comparable to the precision obtained in a single stress test of the same total sample size.

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1. INTRODUCTION AND SUMMARY

The conduct of life tests at more than one level of an environmental factor known to affect the parameters of the life distribution is a common practice. Ordinarily, the environmental factor is set at higher levels of intensity than the test item will meet in service, in order to shorten the expected time to failure. The life test results obtained at these levels are then extrapolated to more usual levels of the environmental factor by fitting constants to a theoretical or empirical relation between the factor levels and one or more parameters of the life distribution. (It is usual to assume the <u>form</u> of the life distribution is not altered by the level of the environmental factor.)

Life testing having these aims, has come to be called accelerated life testing and an extensive literature on the subject has developed, primarily in connection with the testing of electronic components. The environmental or accelerating factor is customarily called stress, but may actually be voltage, load, temperature, etc.

The work described in this paper is applicable to the analysis of censored accelerated life tests in which i) a two parameter Weibull distribution governs at each stress and ii) the Weibull scale parameter varies inversely with a power of the stress while iii) the shape parameter is invariant with stress.

The primary aims of the present work are to find point and interval estimators of i) a quantile of the life distribution at a specified stress, ii) the Weibull shape parameter and iii) the exponent of the stress-life model based upon the method of maximum likelihood. These aims are analogous to the traditional aims of regression analysis.

Singpurwalla [1] has considered the case where the lives follow the single parameter exponential distribution with the scale parameter varying inversely with a power of the stress and where the data at each stress level is censored. He derives the maximum likelihood (ML) estimates of the stress-life exponent and the constant of proportionality of the stress-life law. He uses asymptotic theory for setting confidence limits on these parameters.

Nelson [2] treats the two parameter Weibull case but estimates the parameters separately at each stress. He then estimates the common shape parameter as a weighted combination of the individual shape parameter estimates and the stress-life exponent and proportionality constant by weighted least squares using the logarithmically transformed stress values as the independent variable.

In [3] Singpurwalla and Al-Khayyal under the same assumptions as [2] consider the direct use of the method of maximum likelihood to estimate the common shape parameter and the stress-life exponent and constant and find the asymptotic covariance matrix of the estimators for the uncensored case.

In a numerical example they calculate these estimates by direct maximization of the likelihood function using SUMT.

In Section 2 of the present paper, the ML estimation equations are derived for the joint estimation of the two parameters of the stress-life law and a general quantile of the life distribution at arbitrary stress S for the case where n_i items are tested at each of k stress levels $S_1 \ldots S_k$ and testing continues to the occurrence of the r_i-th failure at each stress.

In Section 3, the ML equations derived in Section 2 are tested to determine whether they reduce to the ordinary equations when a single stress level is employed and whether the estimates are properly invariant when the units of the stress scale are changed.

Section 4 contains the derivation of pivotal functions which, given their distributions for specific sample sizes, will permit bias correction of, and the setting of confidence limits on, 1) a general quantile of the life distribution at stress S, 2) the Weibull shape parameter and 3) the stress-life exponent.

Section 5 contains the description of a scheme found to be effective for the numerical solution of the likelihood equations and briefly describes a computer program for the Monte Carlo calculation of the distribution of the pivotal functions derived in Section 4.

Section 6 is a numerical example of the analysis of a set of rolling contact fatigue data obtained at four levels of the contact stress.

2. PROBLEM FORMULATION AND DERIVATION OF ESTIMATORS

We consider a series of life tests in which a sample is tested at each of k stress levels. We suppose that for $i = 1 \ldots k$, n_i specimens are subjected to stress S_i and run until the occurrence of the first r_i failures.

It is assumed that under stress S_i the time-to-failure of the test item is a two-parameter Weibull random variable having a scale parameter η_i and a stress-independent shape parameter β. That is, the cumulative distribution function of the life at stress S_i is expressible as

$$\text{Prob}[\text{life} < x \mid S = S_i] = 1 - \exp[-(x/\eta_i)^{\beta}] \qquad (1)$$

The scale parameter η_i is assumed to vary inversely with the γ-th power of the stress S_i, i.e. as:

$$\eta_i = \eta_o \, S_i^{-\gamma} \qquad (2)$$

where η_o is a constant representing the scale parameter at unity stress.

We denote by $x_{i(j)}$ the j-th ordered life achieved at stress S_i with $x_{i(r_i)} = x_{i(r_i+1)} = \ldots x_{i(n_i)}$.

Given the results of such a life test, we wish to form the maximum likelihood (ML) estimates of the parameters η_o, β and γ.

The logarithm of the likelihood function written for all k samples is found to have the form

$$\log L = \log c + \log \beta \sum_{i=1}^{k} r_i + (\beta - 1) \sum_{i=1}^{k} \sum_{j=1}^{r_i} \log x_{i(j)} \qquad (3)$$

$$-\beta \sum_{i=1}^{k} r_i \log \eta_i - \sum_{i=1}^{k} \sum_{j=1}^{n_i} [x_{i(j)}/\eta_i]^{\beta}$$

Equating to zero the derivative of logL with respect to η_o gives the following equation which must be satisfied by the ML estimates $\hat{\eta}_o$, $\hat{\beta}$ and $\hat{\gamma}$.

$$\frac{\partial \log L}{\partial \eta_o} = -\hat{\beta}/\hat{\eta}_o \sum_{i=1}^{k} r_i + \hat{\beta}\hat{\eta}_o^{-\hat{\beta}-1} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left(\frac{x_{i(j)}}{S_i^{-\hat{\gamma}}}\right)^{\hat{\beta}} = 0 \tag{4}$$

The solution for $\hat{\eta}_o$ in terms of $\hat{\gamma}$ and $\hat{\beta}$ is:

$$\hat{\eta}_o = \left\{ \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left(\frac{x_{i(j)}}{S_i^{-\hat{\gamma}}}\right)^{\hat{\beta}} \Big/ \sum_{i=1}^{k} r_i \right\}^{1/\hat{\beta}} \tag{5}$$

When $n_i = n$ and $r_i = r$ for all i this equation specializes to:

$$\hat{\eta}_o = \left\{\frac{1}{rk} \sum_{i=1}^{k} \sum_{j=1}^{n} \left(\frac{x_{i(j)}}{S_i^{-\hat{\gamma}}}\right)^{\hat{\beta}}\right\}^{1/\hat{\beta}} \tag{6}$$

Differentiating Eq. (3) with respect to γ and substituting from Eq. (5) gives:

$$\frac{\partial \log L}{\partial \gamma} = \sum_{i=1}^{k} r_i \log S_i - \frac{\sum_{i=1}^{k} r_i \sum_{i=1}^{k} S_i^{\hat{\gamma}\hat{\beta}} \log S_i \sum_{j=1}^{n_i} [x_{i(j)}]^{\hat{\beta}}}{\sum_{i=1}^{k} S_i^{\hat{\gamma}\hat{\beta}} \sum^{n_i} [x_{i(j)}]^{\hat{\beta}}} = 0 \tag{7}$$

For $r_i = r$, $n_i = n$, Eq. (7) reduces to:

$$\frac{\partial \log L}{\partial \gamma} = \sum_{i=1}^{k} \log S_i - \frac{k \sum_{i=1}^{k} S_i^{\hat{\gamma}\hat{\beta}} \log S_i \sum_{j=1}^{n} [x_{i(j)}]^{\hat{\beta}}}{\sum_{i=1}^{k} S_i^{\hat{\gamma}\hat{\beta}} \sum_{j=1}^{n} x_{i(j)}} = 0 \tag{8}$$

Finally, differentiating Eq. (3) with respect to β and using Eqs. (5) and (7) leads, after considerable simplification, to the expression:

$$\frac{1}{\hat{\beta}} + \frac{\sum_{i=1}^{k}\sum_{j=1}^{r_i} \log x_{i(j)}}{\sum_{i=1}^{k} r_i} - \frac{\sum_{i=1}^{k} S_i^{\hat{\gamma}\hat{\beta}} \sum_{j=1}^{n_i} x_{i(j)}^{\hat{\beta}} \log x_{i(j)}}{\sum_{i=1}^{k} S_i^{\hat{\gamma}\hat{\beta}} \sum_{j=1}^{r_i} x_{i(j)}^{\hat{\beta}}} = 0 \qquad (9)$$

For $r_i = r$ and $n_i = n$, Eq. (9) specializes to:

$$\frac{1}{\hat{\beta}} + \frac{1}{rk} \cdot \sum_{i=1}^{k}\sum_{j=1}^{r} \log x_{i(j)} - \frac{\sum_{i=1}^{k} S_i^{\hat{\gamma}\hat{\beta}} \sum_{j=1}^{n} x_{i(j)}^{\hat{\beta}} \log x_{i(j)}}{\sum_{i=1}^{k} S_i^{\hat{\gamma}\hat{\beta}} \sum_{j=1}^{n} x_{i(j)}^{\hat{\beta}}} = 0 \qquad (10)$$

Following the simultaneous solution of Eqs. (7) and (9) for $\hat{\beta}$ and $\hat{\gamma}$ one may evaluate $\hat{\eta}_o$ from Eq. (5). The ML estimate of η_i is obtained using Eq. (2) and the fact that ML estimates of functions of parameters are just those functions of the ML estimates, i.e.

$$\hat{\eta}_i = \hat{\eta}_o \, S_i^{-\hat{\gamma}} \qquad (11)$$

The p-th quantile of the life distribution under stress S_i may be estimated as:

$$\hat{x}_{p_i} = \left[\log\left(\frac{1}{1-p}\right)\right]^{1/\hat{\beta}} \cdot \hat{\eta}_i \qquad (12)$$

3. CHECKING THE VALIDITY OF THE EQUATIONS

As a check on the reasonableness of the likelihood equations given above we consider the special case in which all testing is at a single stress, i.e. $S_i = S$ (all i). In this case the stress-life exponent γ should be unestimable and the equations for estimating β and η_i should reduce to the expression known to apply when a single sample is considered.

From Eq. (8) with $S_i = S$ one has:

$$k\log S - \frac{k\, S^{\hat{\gamma}\hat{\beta}} \log S \sum_{i=1}^{k} \sum_{j=1}^{n} x_{i(j)}^{\hat{\beta}}}{S^{\hat{\gamma}\hat{\beta}} \sum_{i=1}^{k} \sum_{j=1}^{n} x_{i(j)}^{\hat{\beta}}} = 0 \qquad (13)$$

which reduces to:

$$k \log S - k \log S = 0$$

i.e., the equation is satisfied for any value of $\hat{\gamma}$.

Eq. (10) with $S_i = S$ becomes:

$$\frac{1}{\hat{\beta}} + \frac{1}{rk} \sum_{i=1}^{k} \sum_{j=1}^{r} \log x_{i(j)} - \frac{S^{\hat{\gamma}\hat{\beta}} \sum_{i=1}^{k} \sum_{j=1}^{n} x_{i(j)}^{\hat{\beta}} \log x_{i(j)}}{S^{\hat{\gamma}\hat{\beta}} \sum_{i=1}^{k} \sum_{j=1}^{n} x_{i(j)}^{\hat{\beta}}} = 0 \qquad (14)$$

The estimate of η_o becomes, from Eq. (6):

$$\hat{\eta}_o = S^{\hat{\gamma}} \{\frac{1}{rk} \sum_{i=1}^{k} \sum_{j=1}^{n} x_{i(j)}^{\hat{\beta}}\}^{1/\hat{\beta}} \qquad (15)$$

The estimate of η_i, the Weibull scale parameter at stress S using Eq. (11) then becomes:

$$\hat{\eta}_i = \{\frac{1}{rk} \sum_{i=1}^{k} \sum_{j=1}^{n} x_{i(j)}^{\hat{\beta}}\}^{1/\hat{\beta}} \qquad (16)$$

Eqs. (14) and (16) are seen to be equivalent to the ML equations for estimating η and β from a single sample of size kn having rk failures (cf. Cohen [4]).

Another check for reasonableness of the equations can be made by virtue of the fact that the estimates of β, γ and η_i should be invariant with respect to a change in scale of the stress e.g. the estimates should not be affected if the units in which stress is measured are changed from English to metric.

To test whether this is true, introduce a scale change

$$\sigma_i = cS_i \tag{17}$$

where c is a constant and σ_i is a multiplicatively transformed value of S_i.

Substituting $S_i = \sigma_i/c$ into Eq. (8) gives:

$$\sum_{i=1}^{k} \log\sigma_i - k\log c - \frac{k \sum_{i=1}^{k} \left(\frac{\sigma_i}{c}\right)^{\hat{\gamma}\hat{\beta}} [\log\sigma_i - \log c] \sum_{j=1}^{n} x_{i(j)}^{\hat{\beta}}}{\sum_{i=1}^{k} \left(\frac{\sigma_i}{c}\right)^{\hat{\gamma}\hat{\beta}} \sum_{j=1}^{n} x_{i(j)}^{\hat{\beta}}} = 0 \tag{18}$$

which becomes:

$$\sum_{i=1}^{k} \log\sigma_i - k \log c - \frac{k \sum_{i=1}^{k} \sigma_i^{\hat{\gamma}\hat{\beta}} \log\sigma_i}{\sum_{i=1}^{k} \sigma_i^{\hat{\gamma}\hat{\beta}} \sum_{j=1}^{n} x_{i(j)}^{\hat{\beta}}} + k \log c = 0 \tag{19}$$

Substituting $S_i = \sigma_i/c$ into Eq. (10) gives:

$$\frac{1}{\hat{\beta}} + \frac{1}{rk}\sum_{i=1}^{k}\sum_{j=1}^{r}\log x_{i(j)} - \frac{(1/c^{\hat{\gamma}\hat{\beta}})\sum_{i=1}^{k}\sigma_i^{\hat{\gamma}\hat{\beta}}\sum_{j=1}^{n}x_{i(j)}^{\hat{\beta}}\log x_{i(j)}}{(1/c^{\hat{\gamma}\hat{\beta}})\sum_{i=1}^{k}\sigma_i^{\hat{\gamma}\hat{\beta}}\sum_{j=1}^{n}x_{i(j)}^{\hat{\beta}}} = 0 \quad (20)$$

Eqs. (19) and (20) are identical to Eqs. (8) and (10) respectively except that σ_i replaces S_i. Thus the simultaneous solution of Eqs. (8) and (10) for $\hat{\gamma}$ and $\hat{\beta}$ is invariant with respect to a scale change in the stresses.

The solution for $\hat{\eta}_o$ using $S_i = \sigma_i/c$ becomes from Eq. (6):

$$\hat{\eta}_o = c^{-\hat{\gamma}}\left\{\frac{1}{rk}\sum_{i=1}^{k}\sum_{j=1}^{n}\left(\frac{x_{i(j)}}{\sigma_i^{-\hat{\gamma}}}\right)^{\hat{\beta}}\right\}^{1/\hat{\beta}} \quad (21)$$

Eq. (11) then becomes:

$$\hat{\eta}_i = \frac{\sigma_i^{-\hat{\gamma}}}{c^{-\hat{\gamma}}}\; c^{-\hat{\gamma}}\left\{\frac{1}{rk}\sum_{i=1}^{k}\sum_{j=1}^{n}\left(\frac{x_{i(j)}}{\sigma_i^{-\hat{\gamma}}}\right)^{\hat{\beta}}\right\}^{1/\hat{\beta}} \quad (22)$$

Again, Eq. (22) is identical to Eq. (11) with σ_i replacing S_i, so that the ML estimate of η_i is unaltered by a scale change in stress.

4. PIVOTAL FUNCTIONS

It is possible to draw inferences, i.e. set confidence intervals and make hypothesis tests for a parameter if one can determine a function of the parameter and its estimate that follows a distribution that depends on sample size but not upon that parameter or any other parameters of the distribution. Such functions are designated pivotal functions and have been found for the shape parameter and a general quantile in the single sample case of the two parameter Weibull distribution [cf. McCool (5)].

In searching for pivotal functions for the present problem we use the strategy employed in [5] of expressing a Weibull random variable in terms of its population parameters and a rectangular variable. When this is done the order statistic $x_{i(j)}$ transforms to

$$x_{i(j)} = \eta_i \{-\log u_{ij}\}^{1/\beta} \tag{23}$$

where the variables u_{ij} follow beta distributions with parameters that depend on sample size but not upon η_i or β .

For simplicity we hereafter restrict attention to the special case of constant sample size in which $n_i = n$ and $r_i = r$. The results apply to the unequal sample size case as well.

Substituting Eq. (23) into Eq. (8) and using (2) results, after some simplification, in the equation:

$$\sum_{i=1}^{k} \log S_i - \frac{k \sum\limits_{i=1}^{k} S_i^{\hat{\beta}(\hat{\gamma}-\hat{\gamma})} \log S_i \sum\limits_{j=1}^{n} \{-\log u_{ij}\}^{\hat{\beta}/\beta}}{\sum\limits_{i-1}^{k} S_i^{\hat{\beta}(\gamma-\hat{\gamma})} \sum\limits_{j=1}^{n} \{-\log u_{ij}\}^{\hat{\beta}/\beta}} = 0 \tag{24}$$

Substituting Eq. (23) into Eq. (10) and using Eq. (24) leads to:

$$\frac{\hat{\beta}}{\beta}+\frac{1}{rk}\sum_{i=1}^{k}\sum_{j=1}^{r}\log\log(1/u_{ij})-\frac{\sum_{i=1}^{k}\sum_{j=1}^{n} S_i^{\hat{\beta}(\hat{\gamma}-\gamma)}\{-\log u_{ij}\}^{\hat{\beta}/\beta}\log\log(1/u_{ij})}{\sum_{i=1}^{k}\sum_{j=1}^{n} S_i^{\hat{\beta}(\hat{\gamma}-\gamma)}[-\log u_{ij}]^{\hat{\beta}/\beta}} = 0 \quad (25)$$

If one now writes $\hat{\beta}(\hat{\gamma}-\gamma)$ as $(\hat{\beta}/\beta)(\hat{\gamma}-\gamma)\beta$ it is seen that for a given set of u_{ij} Eqs. (24) and (25) can be solved simultaneously for the quantities $q \equiv \hat{\beta}/\beta$ and $s \equiv (\hat{\gamma}-\gamma)\beta$. In repeated sampling i.e. different sets of u_{ij}, these quantities will vary in a manner that depends only on k, r and n.

If for specified values of k, r and n repeated Monte Carlo samples were drawn from a two parameter Weibull population having say $\beta = 1.0$ and $\gamma = 0$, one could empirically determine as closely as desired the distribution of $\hat{\beta}$ and hence of q and the distribution of $\hat{\gamma}$ and hence of s.

Denoting the 100 α-th percentage point of the distribution of q by $q_\alpha(r,n,k)$, one may invert the probability inequality:

$$P\,[q_{0.05}(r,n,k) < \hat{\beta}/\beta < q_{0.95}(r,n,k)] = 0.90 \quad (26)$$

to obtain a 90% confidence interval for β as:

$$\hat{\beta}/q_{0.95}(r,n,k) < \beta < \hat{\beta}/q_{0.05}(r,n,k) \quad (27)$$

A median unbiased estimate of β would be:

$$\hat{\beta}' = \hat{\beta}/q_{0.50}(r,n,k) \quad (28)$$

As proposed in [6] the ratio R of the upper to lower ends of a confidence interval on the shape parameter is a useful measure of the precision with which the shape parameter is determined in a sample of given size.

Arbitrarily using a 90% interval it follows from Eq. (27) that

$$R = q_{0.95}/q_{0.05} \tag{29}$$

For setting confidence intervals on γ it is noted that since the random variables s and q are distributed independently of the Weibull parameters, so is their product $w^* = s \cdot q$, i.e.:

$$w^* = (\hat{\gamma}-\gamma)\beta \cdot \hat{\beta}/\beta = (\hat{\gamma}-\gamma)\hat{\beta} \tag{30}$$

Thus from the distribution of $w^*(r,n,k)$ one may set a 90% confidence interval for γ as:

$$\hat{\gamma} - \frac{w^*_{0.95}\ (r,n,k)}{\hat{\beta}} < \gamma < \hat{\gamma} - \frac{w^*_{0.05}\ (r,n,k)}{\hat{\beta}} \tag{31}$$

A median unbiased estimate of γ is calculable as:

$$\hat{\gamma}' = \hat{\gamma} - \frac{w^*_{0.50}\ (r,n,k)}{\hat{\beta}} \tag{32}$$

A convenient measure of the precision of determination for γ is the median length $L_{0.50}$ of a (say) 90% confidence interval.

From Eq. (35) and the definition of q, $L_{0.50}$ may be calculated as:

$$L_{0.50} = \frac{w^*_{0.95} - w^*_{0.05}}{\beta\ q_{0.50}} \tag{33}$$

$L_{0.50}$ depends on the sample size parameters, i.e. k, n and r and also on the true but unknown value of the shape parameter β.

The ratio of $L_{0.50}$ values for various sample size choices is however independent of β.

Substituting Eq. (23) into Eq. (6) and using Eqs. (11) and (12) gives the following expression for $\hat{x}_{pi}$ in terms of the u_{ij}.

$$\hat{x}_{pi} = S_i^{-\hat{\gamma}} \eta_o \{ \frac{k_p}{rk} \sum_{i=1}^{k} \sum_{j=1}^{n} S_i^{\hat{\beta}(\hat{\gamma}-\gamma)} \{ -\log u_{ij} \}^{\hat{\beta}/\beta} \}^{1/\hat{\beta}} \tag{34}$$

where $k_p \equiv \log(\frac{1}{1-p})$ (35)

The population value of x_{pi} is:

$$x_{pi} = k_p^{1/\beta} S_i^{-\gamma} \eta_o \tag{36}$$

Dividing Eq. (34) by Eq. (36) and raising both sides to the $\hat{\beta}$-th power gives:

$$[\frac{\hat{x}_{pi}}{x_{pi}}]^{\hat{\beta}} = \frac{k_p^{1-\hat{\beta}/\beta}}{rk} S_i^{-\hat{\beta}(\hat{\gamma}-\gamma)} \sum_{i=1}^{k} \sum_{j=1}^{n} S_i^{\hat{\beta}(\hat{\gamma}-\gamma)} \{-\log u_{ij}\}^{\hat{\beta}/\beta} \tag{37}$$

The right hand side of Eq. (37) involves only the u_{ij} and the pivotal functions q and w* and thus the function on the left side of Eq. (37) or its logarithm is a pivotal function.

Defining

$$u^* (r,n,k,p) = \hat{\beta} \log \left[\frac{\hat{x}_{pi}}{x_{pi}}\right] \tag{38}$$

one can, given the percentage points of u^*, set confidence limits on x_{pi}. Two sided 90% confidence limits would have the form:

$$\hat{x}_{pi} \cdot \exp [-u^*_{0.95}/\hat{\beta}] < x_{pi} < \hat{x}_{pi} \cdot \exp [-u^*_{0.05}/\hat{\beta}] \tag{39}$$

A median unbiased estimate of x_{pi} may be calculated as:

$$\hat{x}'_{pi} = \hat{x}_p \cdot \exp [-u^*_{0.50}/\hat{\beta}] \tag{40}$$

It should be noted that all of the foregoing applies even when the stress at which it is desired to estimate x_p is <u>not</u> one of the stresses at which life tests are run. Confidence limits and median unbiased point estimates can be calculated for any stress within or without the range of stresses encompassed by life tests.

A measure proposed in [6] of the precision with which x_{pi} is determined is the median ratio $R_{0.50}$ of the upper to lower ends of its confidence interval. Choosing a 90% interval this ratio becomes, from (39)

$$R_{0.50} = \exp \left[\frac{-u_{0.05} + u_{0.95}}{\beta\, q_{0.50}}\right] \tag{41}$$

wherein the median value of $\hat{\beta}$ is expressed as the product $\beta q_{0.50}$.

5. NUMERICAL SOLUTION OF THE ML EQUATIONS

The solution of Eqs. (9) and (7) by means of a general nonlinear equation solver routine that computes derivatives using finite differences, was found to be slow and unreliably convergent.

The ad hoc solution scheme described below was found to be quite fast and dependable.

Using the transformation

$$y_{ij} \equiv S_i^{\hat{\gamma}} \cdot x_{i(j)} \tag{42}$$

Eq. (9) reduces to

$$\begin{aligned} 1/\hat{\beta} &+ \sum_{i=1}^{k} \sum_{j=1}^{r_i} \log y_{ij} / \sum_{i=1}^{k} r_i - \sum_{i=1}^{k} \sum_{j=1}^{n_i} y_{ij}^{\hat{\beta}} \log y_{ij} / \sum_{i=1}^{k} \sum_{j=1}^{n_i} y_{ij}^{\hat{\beta}} \\ &+ \hat{\gamma} \sum_{i=1}^{k} \sum_{j=1}^{n_i} y_{ij}^{\hat{\beta}} \log S_i / \sum_{i=1}^{k} \sum_{j=1}^{n_i} y_{ij}^{\hat{\beta}} - \hat{\gamma} \sum^{k} r_i \log S_i / \sum_{i=1}^{k} r_i = 0 \end{aligned} \tag{43}$$

while Eq. (7) becomes

$$\sum_{i=1}^{k} r_i \log S_i - \frac{\sum_{i=1}^{k} r_i \sum_{i=1}^{k} \sum_{j=1}^{n_i} y_{ij}^{\hat{\beta}} \log S_i}{\sum_{i=1}^{k} \sum_{j=1}^{n_i} y_{ij}^{\hat{\beta}}} = 0 \tag{44}$$

When Eq. (44) is satisfied the last two terms of Eq. (43) are also satisfied. The remaining terms of Eq. (42) are recognized as the terms that must sum to zero when estimating the Weibull shape parameter from k type II censored data groups with y_{ij}

representing the j-th ordered observation in the i-th group (cf. [7]). For a given set of y_{ij} the solution of these equations for $\hat{\beta}$ is readily accomplished by the Newton-Raphson method.

From this observation the following approach to the simultaneous solution of Eqs. (48) and (49) suggests itself:

1) Guess a value of $\hat{\gamma} = \hat{\gamma}^{(0)}$
2) Transform the data to $y_{ij}^{(0)}$ using $\gamma = \hat{\gamma}$ in Eq. (47)
3) Solve for $\hat{\beta} = \hat{\beta}^{(0)}$ from the first three terms of Eq. (48) using $y_{ij} = y_{ij}^{(0)}$
4) Use $\hat{\beta}^{(0)}$ and $\hat{\gamma}^{(0)}$ in Eq. (49) and solve for an improved $\hat{\gamma} = \hat{\gamma}^{(1)}$ using the Newton-Raphson method.
5) Repeat steps (1)-(4), iteratively replacing $\hat{\gamma}$ by the latest value emerging from step (4), until successive $\hat{\gamma}$ and $\hat{\beta}$ estimates do not differ by more than a prescribed amount.

This procedure was incorporated into a simulation program written in Fortran IV, that generates 10 000 realizations of a random experiment in which k samples of size n are tested at each of k stresses S_1, $S_2 \ldots S_k$ until the first r failures occur.

The failure lives follow a two parameter Weibull distribution with shape parameter $\beta = 1.0$ and with a p-th percentile value x_{pi} at stress $S_i (i=1\ldots k)$.

The input data consists of the values n, r, k, S_i (i=1,k), p and x_{pi} (i=1,k) and the values of three additional stresses S_{k+1}, $S_{k+2} \ldots S_m$ at which it is of interest to calculate x_{pi}.

By setting x_{pi} constant for all i one simulates the case where the stress-life exponent has the value $\gamma = 0$.

The program calculates $\hat{\gamma}$, $\hat{\beta}$ and $\hat{x}_{pi}$ $(i=1,\ldots k+m)$ for each realization and compiles, for specified functions of these observations, their empirical distribution over 10 000 simulated realizations of the experiment.

6. NUMERICAL EXAMPLE

The following data are the ordered times to failure in rolling contact fatigue of ten hardened steel specimens tested at each of four values of the contact stress.

Stress 10^6-psi	Ordered Lives
0.87	1.67, 2.20, 2.51, 3.00, 3.90, 4.70, 7.53, 14.70, 27.76, 37.4
0.99	0.80, 1.00, 1.37, 2.25, 2.95, 3.70, 6.07, 6.65, 7.05, 7.37
1.09	0.012, 0.18, 0.20, 0.24, 0.26, 0.32, 0.32, 0.42, 0.44, 0.88
1.18	0.073, 0.098, 0.117, 0.135, 0.175, 0.262, 0.270, 0.350, 0.386, 0.456

Rolling contact fatigue data of this type are customarily treated as samples from two-parameter Weibull populations having a scale parameter that varies inversely with a power of the contact stress and a shape parameter that is invariant with stress [cf. Lieblein and Zelen (8)].

For the most part these assumptions appear to be satisfied by the present data. Probability plots suggest however that the first failure at the stress S = 1.09 is an outlier and that at S = 0.87 a three parameter Weibull model with a location parameter of about 1.50 is indicated.

For expository purposes we nonetheless accept the two-parameter Weibull model as adequately representing the data.

To test the constant shape parameter assumption we calculate the raw ML shape parameter estimates for each sample, obtaining the values listed below:

Stress (10^6 psi)	0.87	0.99	1.09	1.18
ML Shape Parameter Est.	0.953	1.57	1.43	1.96

Following [9] we employ as test statistic the ratio 1.96/0.953 = 2.06 of the extremal shape parameter estimates.

From values tabled in [9] we find that for $k = 4$, $n = r = 10$, the 90-th percentile of the null distribution of the extremal shape parameter ratio is $(\hat{\beta}_{max}/\hat{\beta}_{min})_{0.90} = 2.47$. Since $2.06 < 2.47$ there is no reason to reject the hypothesis of a common shape parameter.

Proceeding then, we calculate the joint ML estimates of the stress-life exponent γ and the common shape parameter β from Eqs. (8) and (10) to be:

$$\hat{\gamma} = 13.89$$
$$\hat{\beta} = 1.166$$

Additionally we calculate the ML estimate $\hat{\beta}_{(1)}$ of the common shape parameter β that applies when the η_i are not constrained in any way (cf. [7]) with the result $\hat{\beta}_{(1)} = 1.343$.

The estimate $\hat{\beta}_{(1)}$ is closer to the average of the four individual ML estimates tabled above than is the estimate $\hat{\beta} = 1.166$ that results when the scale parameters are constrained to vary with an inverse power of stress. This suggests some lack-of-fit to the inverse power law.

The ML tenth percentile estimates at the four stresses are as follows:

Stress (10^6 psi)		0.87	0.99	1.09	1.18
$\hat{x}_{0.10}$	power law	2.209	0.3672	0.0965	0.0321
	unconstrained	2.328	0.7862	0.0656	0.0457

The simulation program was run with $n = r = 10$, $k = 4$ and $S_1 = 0.87$, $S_2 = 0.99$, $S_3 = 1.09$, $S_4 = 1.18$ and the additional value $S_5 = 0.75$. The distributions were calculated for q, w*, and the u* applicable at each S_i. The 5-th, 50-th and 95-th percentiles of these random variables are listed in Table 1.

Using these tabled values one calculates from Eq. (27) the 90% confidence interval

$$0.913 = 1.166/1.277 < \beta < 1.166/0.8459 = 1.378$$

From Eq. (28) a median unbiased estimate of the shape parameter is

$$\hat{\beta}' = 1.166/1.024 = 1.139$$

From Eq. (29) the precision measure R is calculated as

$$R = \frac{1.277}{0.8459} = 1.51$$

This value is in good agreement with the value calculated for a single uncensored sample of size n = 40, indicating that there is a negligible loss in precision for estimating β by conducting tests at four stresses rather than at only one.

From Eq. (31) a 90% confidence interval for the stress-life parameter is calculated to be

$$11.92 = 13.889 - 2.293/1.166 < \gamma < 13.889 +$$

$$2.433/1.166 = 15.98$$

A median unbiased estimate of γ is from Eq. (32)

$$\hat{\gamma}' = 13.889 + .3783/1.166 = 14.21$$

From Eq. (33) the product of β and the precision measure $L_{0.50}$ is

$$\beta L_{0.50} = \frac{2.293 + 2.433}{1.024} = 4.62$$

Using Eqs. (39), (40) and (41) yields the following values of the median unbiased estimates and 90% confidence limits for $x_{0.10i}$, i = 1,...5. Also listed for each stress are the values of the precision measure $R^{\beta}_{0.50}$.

Stress	Median Unbiased $x_{0.10}$ Estimate	90% Confidence Interval $x_{0.10}$ Lower	Upper	$R^{\beta}_{0.50}$
0.75	16.55	7.22	38.4	6.70
0.87	2.13	1.09	3.86	4.21
0.99	0.358	0.193	0.572	3.42
1.09	0.094	0.050	0.152	3.55
1.18	0.031	0.016	0.054	4.03

Figure 1 shows on logarithmic scales the straight line that joins the raw estimates of $x_{0.10}$ plotted against stress along with the bands formed by the upper and lower confidence limits calculated at each stress.

Also shown are the ML estimates of $x_{0.10}$ at the four stresses at which data were taken computed under the assumption of a common shape parameter but with no constraint on the variation of $x_{0.10}$ with stress and their associated 90% confidence intervals.

The lower confidence limit for the uncontrained estimate at S_2 = 0.99 just touches the line fitted under the power law constraint. This confirms the indication of lack of fit to the power law model suggested by the comparison of the constrained and unconstrained shape parameter estimates discussed earlier.

Fig. (2) is a plot of $R^{\beta}_{0.50}$ against stress level. Fig. (2) illustrates how $R^{\beta}_{0.50}$ is a minimum near the midpoint of the stress range, increasing substantially when extrapolating to stresses outside the range of the tests.

The value of $R_{0.50}^{\beta}$ for a single test of size n =r = 40 was found by simulation to be $R_{0.50}^{\beta}$ = 3.41. This compares favorably with the minimum value shown on Fig. 2. This indicates that if high precision is required at some specific stress level a negligible loss in precision is suffered if, rather than conducting all tests at this stress level, some specimens are tested at surrounding stress levels. Plots like Fig. (2) can be constructed in advance of any actual testing to determine the choice of stress levels, the number of levels and the sample sizes that yield a precision "profile" that the experimenter finds suitable. In addition as noted previously the sample size and censoring number can differ at each stress level.

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TABLE 1

Percentiles of Pivotal Functions

$k = 4,\ n = r = 10$

$S_1=0.87,\ S_2=0.99,\ S_3=1.09,\ S_4=1.18$

	0.05	0.50	0.95
$q = \hat{\beta}/\beta$	0.8459	1.024	1.277
$w^* = (\hat{\gamma}-\gamma)\hat{\beta}$	-2.433	-0.3783	2.293
$u^* = \hat{\beta}\log[\hat{x}_{0.10}/x_{0.10}]$;			
$S = 0.75$	-0.9238	0.0555	1.023
$S = S_1 = 0.87$	-0.6495	0.0441	0.8209
$S = S_2 = 0.99$	-0.5170	0.0305	0.7520
$S = S_3 = 1.09$	-0.5309	0.0318	0.7671
$S = S_4 = 1.18$	-0.6079	0.0396	0.8193

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FIGURE 1

ESTIMATED REGRESSION LINE AND 90% CONFIDENCE BAND

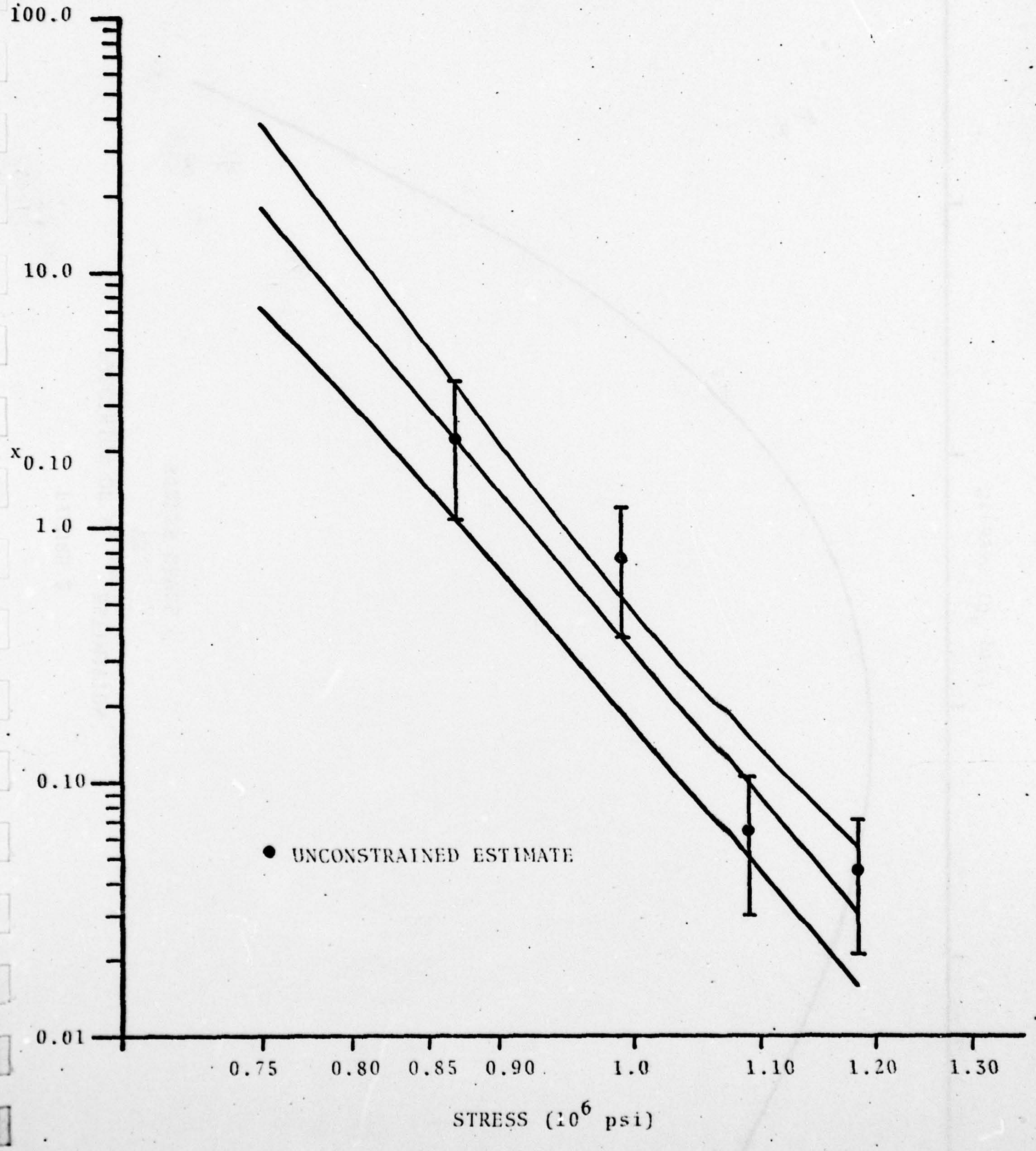

FIGURE 2

PRECISION OF $x_{0.10}$ ESTIMATION
VS.
STRESS LEVEL